MINE SUPPORT MECHANISMS USING A LIMIT EQUILIBRIUM ANALYSIS

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Abstract

The problem posed was to determine the stress distribution and plastic flow in a two-dimensional pillar region in limit equilibrium. It is assumed that the equations of static equilibrium are satisfied in the failed rock region of the pillar and that the Mohr-Coulomb failure criterion is the constitutive equation of the failed rock. An outline of the derivation of the Mohr-Coulomb failure criterion for rock in limit equilibrium is first given. A second order nonlinear partial differential equation for the Airy stress function in a two-dimensional region in limit equilibrium is derived. The partial differential equation admits a similarity solution but the boundary conditions for the pillar cannot be expressed in terms of the similarity variables. It is suggested that the similarity transformation may be applicable in triangular wedge regions in limit equilibrium, such as near the corners where the pillar meets the floor and the roof.

1 Introduction

The failed or broken rock region near excavations and pillar edges cannot be assumed to behave elastically. An approximation that can be made is to represent

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the fractured rock as a pseudo-continuum in which the constitutive properties are assumed to be in a state of limit equilibrium. In this case a fixed constraint exists between the local stress components at each point of the rock mass. A simple example is the Mohr-Coulomb failure criterion.

The problem posed to the Study Group was to determine the detailed stress distribution and plastic flow movements in a two-dimensional region of rock in limit equilibrium. We assume that the damaged region is confined to the pillar region between the footwall (floor) and the hangingwall (roof). We will assume that the equations of static equilibrium are satisfied in the failed rock region. We will take as the constitutive equation of the failed rock the Mohr-Coulomb failure criterion. This criterion is a relation between the maximum and minimum principal stress components and the inherent stress of the rock.

An outline of the report is as follows. In Section 2 the sign convention for stress used in rock mechanics is explained and the equations of static equilibrium are stated. An outline of the derivation of the Mohr-Coulomb failure criterion is given in Section 3. In Section 4 the problem is formulated in terms of the Airy stress function and a nonlinear partial differential equation for the Airy stress function in a two-dimensional region in limit equilibrium is derived. In Section 5 a similarity solution of the partial differential equation and boundary conditions for a pillar in limit equilibrium is investigated. Finally, conclusions are summarised and future work is discussed in Section 6.

2   Equations of static equilibrium

We will adopt the sign convention used in rock mechanics that forces are recorded positive when compressive [1, 2]. This is opposite to the convention used in elasticity and continuum mechanics in which stresses are usually recorded positive when tensile. This change of sign convention leaves all formulae unaltered but all signs have to be changed. The direction of the normal vector $\boldsymbol{n}$ and the stress vector $\boldsymbol{t}(\boldsymbol{n})$ acting on an internal surface element and the components, τ_{ik}, of the Cauchy stress tensor acting on a square element in the xz-plane are shown in Figure 1.

Consider a two-dimensional region of damaged rock which is infinite in the y-direction. Then all quantities are independent of y. We assume that in the damaged region the equations of static equilibrium are satisfied:

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{zx}}{\partial z} = 0 \ , \tag{2.1}$$
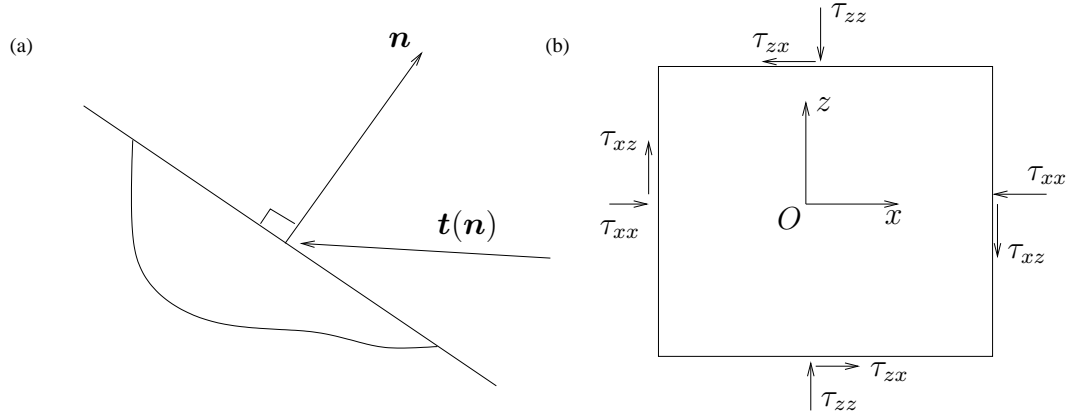
Figure 1: (a) The normal vector $\boldsymbol{n}$ and the stress vector $\boldsymbol{t}(\boldsymbol{n})$ acting on an internal surface element in the rock. (b) The components of the stress tensor acting on a square element in the xz-plane. (Adapted from Jaeger et al. [2].)

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z} = 0 \; . \tag{2.2}$$

Equations (2.1) and (2.2) are two equations for the three unknowns $\tau_{xx}(x, z)$, $\tau_{xz}(x, z)$ and $\tau_{zz}(x, z)$. One further equation relating the three unknowns is required. Since the fractured rock is inelastic, Hooke's law does not apply. We will take for the third equation the Mohr-Coulomb failure criterion for rock in limit equilibrium.

3 Mohr-Coulomb failure criterion

The Mohr-Coulomb failure criterion is obtained by expressing the Coulomb failure criterion in terms of invariants, [1, 2].

Consider first the Coulomb failure criterion. Failure will not occur along an internal plane in a rock mass if

$$\mid \tau \mid < S_0 + \mu\sigma \tag{3.1}$$

where τ is the shear stress and σ is the normal stress on the plane, μ is the coefficient of internal friction of the material and S_0 is the cohesion of the material

which is the finite shear stress still needed to initiate failure when $\sigma = 0$. Failure will occur along the plane if

$$| \tau |= S_0 + \mu\sigma . \tag{3.2}$$

The sign of τ only affects the direction of sliding after failure and therefore only $| \tau |$ appears in (3.2). Equation (3.2) is the Coulomb failure criterion.

The Coulomb failure criterion is expressed in terms of invariants by introducing the principal stresses. The principal stresses are invariants because they are the eigenvalues of the stress tensor. Consider a triangular element of rock as shown in Figure 2. The components of the unit outward normal vector $\boldsymbol{n}$ and the unit tangent vector $\boldsymbol{s}$ to the sloping face are

$$n = (\cos\theta,\ 0, \sin\theta) , \qquad \boldsymbol{s} = (-\sin\theta,\ 0, \cos\theta) . \tag{3.3}$$

From Cauchy's formula, the stress vector, $\boldsymbol{t}(n)$, on the surface with outward normal $\boldsymbol{n}$ is

$$t_i(\boldsymbol{n}) = n_k\tau_{ki} . \tag{3.4}$$

The normal stress acting on the sloping face is σ and the tangential stress is τ. Thus

$$\sigma = \boldsymbol{n} \cdot \boldsymbol{t}(\boldsymbol{n}) = n_in_k\tau_{ki} , \tag{3.5}$$

$$\tau = \boldsymbol{s} \cdot \boldsymbol{t}(\boldsymbol{n}) = s_in_k\tau_{ki} , \tag{3.6}$$

which may be expanded in terms of the components of the stress tensor as

$$\sigma = \frac{1}{2}(\tau_{xx} + \tau_{zz}) + \frac{1}{2}(\tau_{xx} - \tau_{zz})\cos 2\theta + \tau_{xz}\sin 2\theta , \tag{3.7}$$

$$\tau = \frac{1}{2}(\tau_{zz} - \tau_{xx})\sin 2\theta + \tau_{xz}\cos 2\theta . \tag{3.8}$$

The planes on which the shear stress, τ, vanishes satisfy

$$\tan 2\theta = \frac{2\tau_{xz}}{\tau_{xx} - \tau_{zz}} . \tag{3.9}$$

There are two solutions of (3.9) for θ in the range $0 \leq \theta \leq \pi$ and these two solutions differ by $\pi/2$. On these planes

$$\cos 2\theta = \pm\frac{(\tau_{xx} - \tau_{zz})}{\left[(\tau_{xx} - \tau_{zz})^2 + 4\tau_{xz}^2\right]^{1/2}} , \tag{3.10}$$

$$\sin 2\theta = \pm\frac{2\tau_{xz}}{\left[(\tau_{xx} - \tau_{zz})^2 + 4\tau_{xz}^2\right]^{1/2}} \tag{3.11}$$
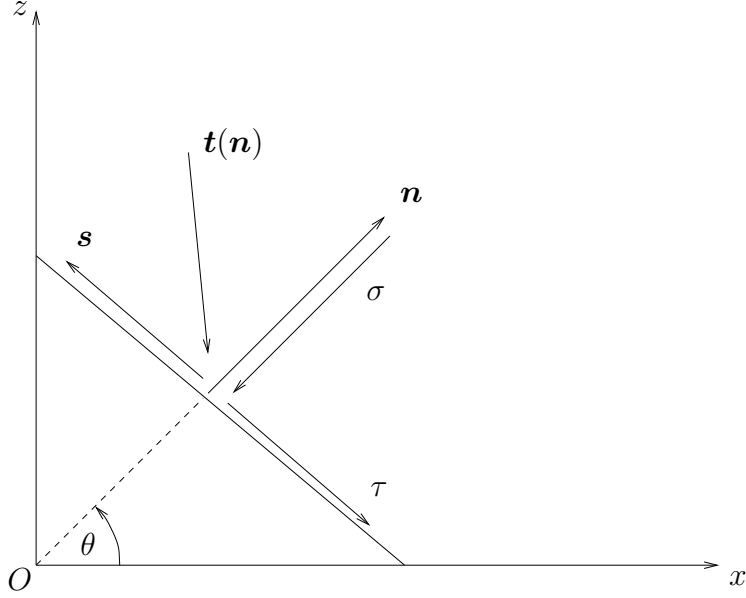
Figure 2: Triangular element of rock. (Adapted from Jaeger et al. [2].)

and the normal stresses are given by

$$\sigma_1 = \frac{1}{2}\left(\tau_{xx} + \tau_{zz}\right) + \frac{1}{2}\left[\left(\tau_{xx} - \tau_{zz}\right)^2 + 4\tau_{xz}^2\right]^{1/2}, \tag{3.12}$$

$$\sigma_3 = \frac{1}{2}\left(\tau_{xx} + \tau_{zz}\right) - \frac{1}{2}\left[\left(\tau_{xx} - \tau_{zz}\right)^2 + 4\tau_{xz}^2\right]^{1/2}. \tag{3.13}$$

The normal stresses, σ_1 and σ_3, are the principal stresses at that point. Since τ_{xx} and τ_{zz} are positive when compressive and the rock mass is generally under compression, it follows that $\sigma_1 > \sigma_3$.

The principal stresses, σ_1 and σ_3, have several important properties. They are the maximum and minimum normal stresses that act on any plane through that point. For, from (3.7) and (3.8),

$$\frac{d\sigma}{d\theta} = \frac{1}{2}\tau \tag{3.14}$$

and therefore on any plane through that point on which $\tau = 0$, the normal stress σ takes a local maximum or minimum value. Thus σ_1 is the maximum and σ_3 is the minimum normal stress on any plane through that point. Also

$$\frac{d\tau}{d\theta} = 0 \tag{3.15}$$

on the planes

$$\tan 2\theta = \frac{\tau_{zz} - \tau_{xx}}{2\tau_{xz}} \ . \tag{3.16}$$

On these planes

$$\cos 2\theta \ = \ \pm \frac{2\tau_{xz}}{\left[(\tau_{xz} - \tau_{xx})^2 + 4\tau_{xz}^2\right]^{1/2}} \ , \tag{3.17}$$

$$\sin 2\theta \ = \ \pm \frac{(\tau_{zz} - \tau_{xx})}{\left[(\tau_{zz} - \tau_{xx})^2 + 4\tau_{xz}^2\right]^{1/2}} \ . \tag{3.18}$$

The maximum shear stress has magnitude

$$\tau_{\max} = \frac{1}{2}\left[(\tau_{zz} - \tau_{xx})^2 + 4\tau_{xz}^2\right]^{1/2} \tag{3.19}$$

and therefore

$$\tau_{\max} = \frac{1}{2}\left(\sigma_1 - \sigma_3\right) \ . \tag{3.20}$$

The minimum magnitude of the shear stress is zero.

Mohr's circle can be used to represent the state of stress at a point and to write the Coulomb failure criterion in terms of the principal stresses. The principal axes of stress are orthogonal. Consider the principal coordinate system in which the x-axis is normal to the plane on which σ_1 acts and the z-axis is normal to the plane on which σ_3 acts. With respect to these axes, $\tau_{xx} = \sigma_1$, $\tau_{zz} = \sigma_3$ and $\tau_{xz} = 0$. Equations (3.7) and (3.8) become

$$\sigma \ = \ \frac{1}{2}\left(\sigma_1 + \sigma_3\right) + \frac{1}{2}\left(\sigma_1 - \sigma_3\right)\cos 2\theta \ , \tag{3.21}$$

$$\tau \ = \ -\frac{1}{2}\left(\sigma_1 - \sigma_3\right)\sin 2\theta \ , \tag{3.22}$$

where θ is the angle of rotation, in the counterclockwise direction, as shown in Figure 2. Eliminating θ between (3.21) and (3.22) gives

$$\left[\sigma - \frac{1}{2}\left(\sigma_1 + \sigma_3\right)\right]^2 + \tau^2 = \frac{1}{4}\left(\sigma_1 - \sigma_3\right)^2 \ . \tag{3.23}$$

Equation (3.23) is the Mohr circle. Its centre is the point $\left[\frac{1}{2}\left(\sigma_1 + \sigma_3\right), 0\right]$ on the x-axis and its radius is $\frac{1}{2}\left(\sigma_1 - \sigma_3\right)$ which from (3.20) is the maximum shear stress on any plane through the point. A point on the Mohr circle rotated clockwise

through an angle 2θ from the σ-axis gives the stress (σ, τ) on the plane whose normal vector is rotated counterclockwise through an angle θ from the maximum principal axis. The Mohr circle is illustrated in Figure 3.

Equation (3.2), written in terms of τ instead of the absolute value $| \tau |$, is

$$\tau = S_0 + \mu\sigma \tag{3.24}$$

which is a straight line in the (σ, τ) plane that makes an angle $\phi = \tan^{-1}\mu$ with the σ-axis and intersects the σ-axis at the point $(-S_0/\mu , 0)$. It is represented in Figure 4 as the straight line APL and is referred to as the Coulomb line or the failure line. From (3.1), a state of stress (σ, τ) which lies below the Coulomb line will not give rise to shear failure on any plane through the point. If the principal stresses, σ_1 and σ_3, are such that the Mohr circle touches the Coulomb line the rock mass will fail in shear. Mohr circles that extend above the Coulomb line have no meaning because failure will occur as soon as the Mohr circle first touches the line.

Since the failure criterion (3.2) involves the absolute value $| \tau |$ the Coulomb line rotated clockwise by an angle ϕ from the σ-axis must also represent a failure plane. There are two possible planes of shear failure making angles β and $-\beta$ with the direction of σ_1 where from Figure 4,

$$2\beta = \frac{\pi}{2} + \phi \ . \tag{3.25}$$

We now express the failure criterion directly in terms of the principal stresses, σ_1 and σ_3. From Figure 4,

$$\sin\phi = \frac{PC}{AO + OC} = \frac{\frac{1}{2}(\sigma_1 - \sigma_3)}{\frac{S_0}{\mu} + \frac{1}{2}(\sigma_1 + \sigma_3)} \tag{3.26}$$

and therefore, using also $\mu = \tan\phi$,

$$\sigma_1 = \frac{2\cos\phi}{1 - \sin\phi} S_0 + \left(\frac{1 + \sin\phi}{1 - \sin\phi}\right)\sigma_3 \ . \tag{3.27}$$

Equation (3.27) is the Mohr-Coulomb failure criterion. We now express it in equivalent forms in terms of β and μ.

To rewrite (3.27) in terms of β we observe that, from (3.25),

$$\cos\phi = \sin 2\beta \ , \qquad \sin\phi = -\cos 2\beta \ . \tag{3.28}$$
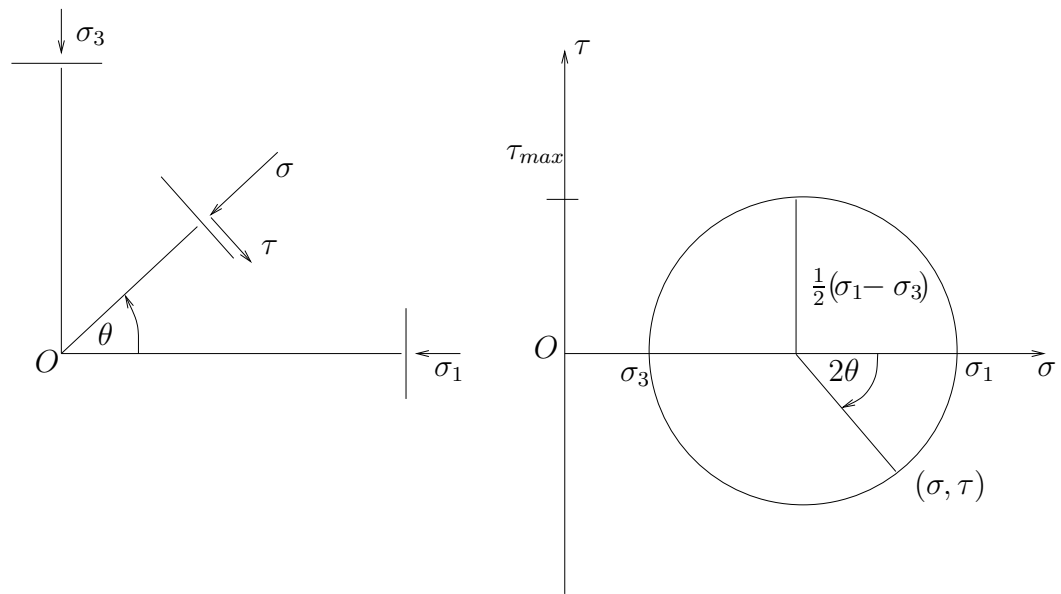
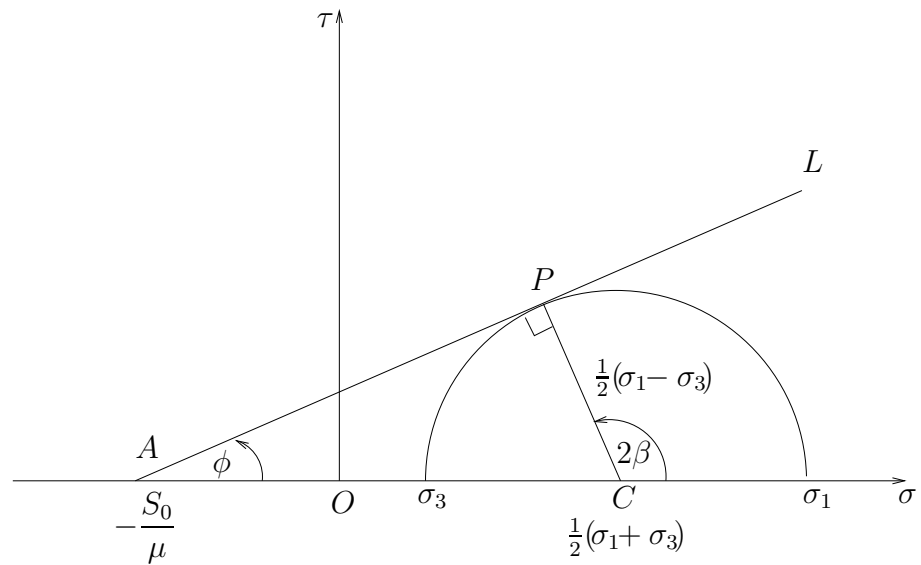Figure 3: The Mohr circle. (Adapted from Jaeger et al. [2].)

Figure 4: Mohr circle and failure line APL. (Adapted from Jaeger et al. [2].)

Equation (3.27) takes the form

$$\sigma_1 = 2S_0 \tan \beta + \sigma_3 \tan^2 \beta . \tag{3.29}$$

Since $\phi \geq 0$, it follows that $\tan \beta \geq 1$. Equation (3.29) can be expressed as

$$\sigma_1 = C_0 + \sigma_3 \tan^2 \beta \tag{3.30}$$

where

$$C_0 = 2S_0 \tan \beta \tag{3.31}$$

is the uniaxial compressive strength.

To rewrite (3.27) in terms of the coefficient of internal friction μ we note that

$$\tan \phi = \mu , \quad \cos \phi = \frac{1}{(1 + \mu^2)^{1/2}} , \quad \sin \phi = \frac{\mu}{(1 + \mu^2)^{1/2}} \tag{3.32}$$

and therefore

$$\sigma_1 = 2\left[(1 + \mu^2)^{1/2} + \mu\right] S_0 + \left[(1 + \mu^2)^{1/2} + \mu\right]^2 \sigma_3 . \tag{3.33}$$

Equation (3.33) can also be derived from (3.29) using

$$\tan \beta = (1 + \mu^2)^{1/2} + \mu . \tag{3.34}$$

Expressed in terms of the uniaxial compressive strength, C_0, (3.33) becomes

$$\sigma_1 = C_0 + \left[(1 + \mu^2)^{1/2} + \mu\right]^2 \sigma_3 \tag{3.35}$$

where

$$C_0 = 2\left[(1 + \mu^2)^{1/2} + \mu\right] S_0 . \tag{3.36}$$

The equivalent forms of the Mohr-Coulomb failure criterion are summarised in Table 1. The magnitude of the intermediate principal stress, σ_2, has no effect on failure.

The rock mass may have a pre-existing plane of weakness. If there are no special planes of weakness all possible planes may be equally weak due to random microcracks. The rock mass will then choose its own plane of failure according to a failure criterion such as the Mohr-Coulomb failure criterion.

Mohr-Coulomb failure criterion	Identities
$\sigma_1 = \dfrac{2\cos\phi}{1-\sin\phi}\, S_0 + \left(\dfrac{1+\sin\phi}{1-\sin\phi}\right)\sigma_3$	$\tan\phi = \mu$
$\sigma_1 = 2S_0\tan\beta + \sigma_3\tan^2\beta$	$\tan\beta = (1+\mu^2)^{1/2} + \mu$
$\sigma_1 = 2\left[(1+\mu^2)^{1/2} + \mu\right]S_0$ $\qquad + \left[(1+\mu^2)^{1/2} + \mu\right]^2 \sigma_3$	
$\sigma_1 = C_0 + \sigma_3\tan^2\beta$	$C_0 = 2S_0\tan\beta$
$\sigma_1 = C_0 + \left[(1+\mu^2)^{1/2} + \mu\right]^2\sigma_3$	$C_0 = 2\left[(1+\mu^2)^{1/2} + \mu\right]S_0$

Table 1: Equivalent forms of the Mohr-Coulomb failure criterion. The principal stresses, σ_1 and σ_3, are related to the components of the Cauchy stress tensor, τ_{xx}, τ_{zz} and τ_{xz}, by (3.12) and (3.13).

4 Stress in a pillar in limit equilibrium and the Airy stress function

The pillars supporting an excavation undergo edge spalling and damage. It will be assumed that the damaged region is confined to the pillar region between the footwall and the hangingwall. The problem is to determine the state of stress in the pillar region.

An illustration of the excavation showing the mined regions, the failed pillar region and the elastic regions is presented in Figure 5. The mined region extends to the left from $x = 0$ to $x = -\infty$ and to the right from $x = L$ to $x = +\infty$ and is bounded by the two planes, $z = -H$ representing the floor and $z = H$ representing the roof of the excavation. The y-axis points into the plane of the diagram. The pillar is infinite in extent in the y-direction and all quantities are independent of y. All the rock in the regions $z < -H$ and $z > H$ is assumed

to be elastic. The failed rock in the pillar is assumed to be in a state of limit equilibrium.

The body force due to gravity is neglected. The non-zero components of the Cauchy stress tensor are $\tau_{xx}(x, z)$, $\tau_{xz}(x, z)$ and $\tau_{zz}(x, z)$. It is assumed that the equations of static equilibrium, (2.1) and (2.2), are satisfied in the failed rock region of the pillar. The third condition is the constitutive equation for rock in the state of limit equilibrium which we take to be the Mohr-Coulomb failure criterion. From Table 1, last entry,

$$\sigma_1 = C_0 + m \; \sigma_3 \; , \tag{4.1}$$

where C_0 is the uniaxial compressive strength,

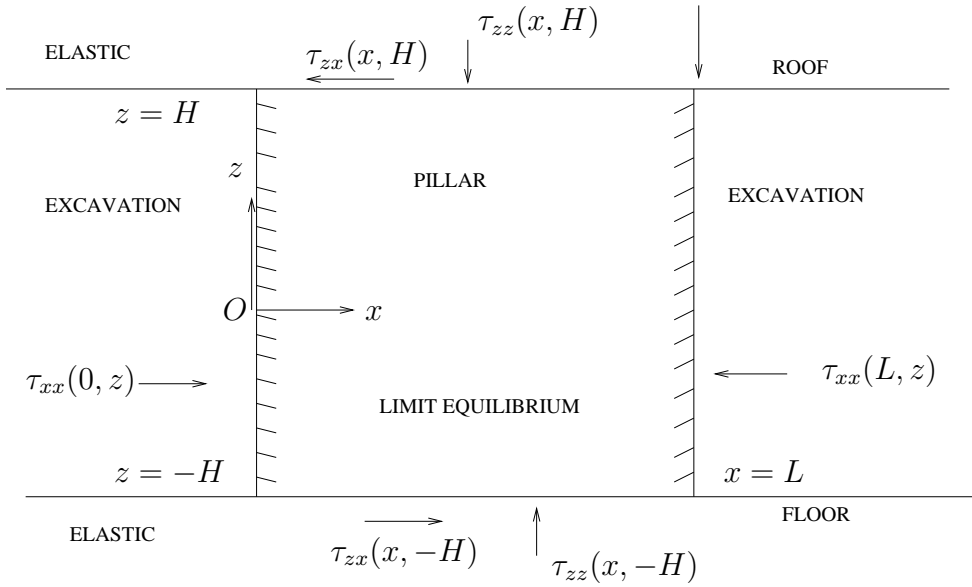$$m = \left[(1 + \mu^2)^{1/2} + \mu \right]^2 > 1 \; , \tag{4.2}$$

Figure 5: Diagram of the excavation showing the mined regions, the pillar in limit equilibrium with length L and height 2H and the elastic regions.

where μ is the coefficient of internal friction of the pillar and σ_1 and σ_3 are the principal stresses at any given point in the pillar with $\sigma_1 > \sigma_3$. Now σ_1 and σ_3

are given by (3.12) and (3.13) and substituting these expressions into (4.1) puts the Mohr-Colombo failure criterion in the form

$$(m-1)(\tau_{xx} + \tau_{zz}) - (m+1)\left[(\tau_{xx} - \tau_{zz})^2 + 4\tau_{xz}^2\right]^{1/2} = -2C_0 . \qquad (4.3)$$

We will formulate the problem in terms of the Airy stress function which replaces the three equations, (2.1), (2.2) and (4.3) by one partial differential equation. By defining the Airy stress function $U(x, z)$ by [1, 2],

$$\tau_{xx}(x, z) = \frac{\partial^2 U}{\partial z^2} , \qquad \tau_{xz}(x, z) = -\frac{\partial^2 U}{\partial x \partial z} , \qquad \tau_{zz}(x, z) = \frac{\partial^2 U}{\partial x^2} , \qquad (4.4)$$

we ensure that the equations of static equilibrium, (2.1) and (2.2), are satisfied and the Mohr-Coulomb failure criterion (4.3) gives

$$(m-1)\left[\frac{\partial^2 U}{\partial z^2} + \frac{\partial^2 U}{\partial x^2}\right] - (m+1)\left[\left(\frac{\partial^2 U}{\partial z^2} - \frac{\partial^2 U}{\partial x^2}\right)^2 + 4\left(\frac{\partial^2 U}{\partial x \partial z}\right)^2\right]^{1/2} = -2C_0 ,$$
$$(4.5)$$

a single field equation for $U(x, z)$. Equation (4.5) has been derived for $m = 1$ by starting from the von Mises yield criterion [1, 2].

Equation (4.5) may be written in terms of dimensionless variables by making the transformations

$$x^* = \frac{x}{L} , \qquad z^* = \frac{z}{H} , \qquad U^* = \frac{U}{C_0 H^2} . \qquad (4.6)$$

Suppressing the star, (4.5) becomes

$$(m-1)\left[\frac{\partial^2 U}{\partial z^2} + \left(\frac{H}{L}\right)^2 \frac{\partial^2 U}{\partial x^2}\right]$$

$$-(m+1)\left[\left(\frac{\partial^2 U}{\partial z^2} - \left(\frac{H}{L}\right)^2 \frac{\partial^2 U}{\partial x^2}\right)^2 + 4\left(\frac{H}{L}\right)^2\left(\frac{\partial^2 U}{\partial x \partial z}\right)^2\right]^{1/2} = -2 . \quad (4.7)$$

The boundary conditions are:

$$x = 0: \quad \tau_{xx}(0, z) = 0 \,, \qquad \frac{\partial^2 U}{\partial z^2}(0, z) = 0 \,, \qquad (4.8)$$

$$x = 0: \quad \tau_{zx}(0, z) = 0 \,, \qquad \frac{\partial^2 U}{\partial x \partial z}(0, z) = 0 \,, \qquad (4.9)$$

$$x = 1: \quad \tau_{xx}(1, z) = 0 \,, \qquad \frac{\partial^2 U}{\partial z^2}(1, z) = 0 \,, \qquad (4.10)$$

$$x = 1: \quad \tau_{zx}(1, z) = 0 \,, \qquad \frac{\partial^2 U}{\partial x \partial z}(1, z) = 0 \,, \qquad (4.11)$$

$$z = 1: \quad \tau_{xz}(x, 1) = -\mu \tau_{zz}(x, 1) \,, \frac{\partial^2 U}{\partial x \partial z}(x, 1) = \mu \frac{\partial^2 U}{\partial x^2}(x, 1) \,, \qquad (4.12)$$

$$z = -1: \quad \tau_{xz}(x, -1) = +\mu \tau_{zz}(x, -1) \,, \frac{\partial^2 U}{\partial x \partial z}(x, -1) = -\mu \frac{\partial^2 U}{\partial x^2}(x, -1) \,. \qquad (4.13)$$

The boundary conditions (4.12) and (4.13) are in the form of limiting equilibrium.

The solution to the problem could be investigated in several ways, for example, numerically or by making a perturbation expansion in H/L if $H \ll L$. We will investigate if the problem admits a similarity solution.

5   Scaling transformation

The method described by Dresner [3] for the derivation of similarity solutions will be used.

Consider the scaling transformation

$$\bar{x} = \lambda^a x \,, \qquad \bar{z} = \lambda^b z \,, \qquad \overline{U} = \lambda^c U \,. \qquad (5.1)$$

The partial differential equation (4.7) becomes

$$(m-1)\left[\lambda^{2b-2a}\frac{\partial^2\overline{U}}{\partial\bar{z}^2}+\left(\frac{H}{L}\right)^2\frac{\partial^2\overline{U}}{\partial\bar{x}^2}\right]$$

$$-(m+1)\left[\left(\lambda^{2b-2a}\frac{\partial^2\overline{U}}{\partial\bar{z}^2}-\left(\frac{H}{L}\right)^2\frac{\partial^2\overline{U}}{\partial\bar{x}^2}\right)^2+4\lambda^{2b-2a}\left(\frac{H}{L}\right)^2\left(\frac{\partial^2\overline{U}}{\partial\bar{x}\partial\bar{z}}\right)^2\right]^{1/2}$$

$$=-2\lambda^{c-2a}\ . \tag{5.2}$$

Now, (4.7) is invariant under the scaling transformation (5.1) if

$$(m-1)\left[\frac{\partial^2\overline{U}}{\partial\bar{z}^2}+\left(\frac{H}{L}\right)^2\frac{\partial^2\overline{U}}{\partial\bar{x}^2}\right]$$

$$-(m+1)\left[\left(\frac{\partial^2\overline{U}}{\partial\bar{z}^2}-\left(\frac{H}{L}\right)^2\frac{\partial^2\overline{U}}{\partial\bar{x}^2}\right)^2+4\left(\frac{H}{L}\right)^2\left(\frac{\partial^2\overline{U}}{\partial\bar{x}\partial\bar{z}}\right)^2\right]^{1/2}=-2\ , \tag{5.3}$$

that is, if $b=a$ and $c=2a$. Hence (4.7) is invariant under the scaling transformation

$$\bar{x}=\lambda^a x\ , \qquad \bar{z}=\lambda^a z\ , \qquad \overline{U}=\lambda^{2a}U\ . \tag{5.4}$$

Suppose that the solution of (4.7) is

$$U=f(x,z)\ . \tag{5.5}$$

Then the solution of (5.3) is

$$\overline{U}=f(\bar{x},\bar{z})\ . \tag{5.6}$$

Thus using (5.4),

$$\lambda^{2a}f(x,z)=f(\lambda^a x\ ,\lambda^a z)\ . \tag{5.7}$$

Differentiate (5.7) with respect to λ and then set $\lambda=1$. This gives

$$x\ \frac{\partial f}{\partial x}+z\ \frac{\partial f}{\partial z}=2f\ . \tag{5.8}$$

The general solution of (5.8) is

$$f(x,z)=x^2 F\left(\frac{z}{x}\right)\ , \tag{5.9}$$

where F is an arbitrary function. Hence

$$U(x,z)=x^2 F(\xi)\ , \qquad \xi=\frac{z}{x}\ . \tag{5.10}$$

Substitution of (5.10) into (4.7) gives the ordinary differential equation

$$
\begin{aligned}
(m-1) & \left[\frac{d^2 F}{d\xi^2} + \left(\frac{H}{L}\right)^2 \left(\xi^2 \frac{d^2 F}{d\xi^2} - 2\xi \, \frac{dF}{d\xi} + 2F\right)\right] \\
-(m+1) & \left[\left(\frac{d^2 F}{d\xi^2} - \left(\frac{H}{L}\right)^2 \left(\xi^2 \frac{d^2 F}{d\xi^2} - 2\xi \, \frac{dF}{d\xi} + 2F\right)\right)^2 \right. \\
& \left. +4 \left(\frac{H}{L}\right)^2 \left(\xi \frac{d^2 F}{d\xi^2} - \frac{dF}{d\xi}\right)^2 \right]^{1/2} = -2 \,.
\end{aligned} \tag{5.11}
$$

The components of the Cauchy stress tensor are

$$
\tau_{xx} = \frac{d^2 F}{d\xi^2} \,, \tag{5.12}
$$

$$
\tau_{xz} = \frac{H}{L} \left(\xi \frac{d^2 F}{d\xi^2} - \frac{dF}{d\xi}\right) \,, \tag{5.13}
$$

$$
\tau_{zz} = \left(\frac{H}{L}\right)^2 \left(\xi^2 \frac{d^2 F}{d\xi^2} - 2\xi \, \frac{dF}{d\xi} + 2F\right) \,. \tag{5.14}
$$

Although the partial differential equation and the components of the stress tensor can be expressed in similarity form the boundary conditions cannot be expressed in terms of the similarity variables and a similarity solution does not exist. The similarity transformation (5.10) may be useful in other problems, for example, in a region with a sloping interface on which z/x is constant.

6   Conclusions and future work

We have assumed that the Mohr-Coulomb failure criterion holds throughout the pillar. There are critical regions of rock in limit equilibrium in the pillar, for example, near where the pillar meets the footwall and the hanging wall and if these regions extend to the full pillar then the Mohr-Coulomb failure criterion will apply throughout the pillar. If they do not extend to the whole pillar there will be regions in limit equilibrium adjacent to elastic or rigid zones.

The Airy stress function formulation gave rise to a second order nonlinear partial differential equation. The partial differential equation admitted a similarity solution but the boundary conditions for the problem of the pillar could not be written in terms of the similarity variables. These variables therefore could not be used in the full pillar problem. The similarity solution, however, may be useful in triangular regions of failed rock in corners where the pillar meets the footwall or hanging wall when criticality does not extend to the whole pillar.

Future work could be to investigate the numerical solution of the partial differential equation for the Airy stress function (4.7) subject to the boundary conditions for a pillar, (4.8) to (4.13). This boundary value problem may be in a form suitable for numerical investigation. The analysis presented here has to be extended in future studies to delineate the regions within the pillar where the material is in a state of limit equilibrium and the regions where the pillar material remains intact. We isolated the pillar itself and assumed that the surrounding material does not fail. This may of course not be the case but that should also be treated as an extended problem.

The Mohr-Coulomb failure criterion (4.1) predicts that σ_1 will increase linearly with σ_3. Experimental results show that σ_1 will generally increase with σ_3 at a rate that is less than linear [2]. Many failure criterion of the form $\sigma_1 = f(\sigma_3)$ containing two or more adjustable parameters have been proposed. A failure criterion which is widely used is the Hoek-Brown criterion which can fit experimental data from many rocks [2, ?]. The mathematical formulation in terms of the Airy stress function could also be developed for other failure criterion.

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